Flow Identities of the Infinite and Bounded Young's Lattice

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Abstract

We describe and prove several new identities of Young's lattice, a partially ordered set of integer partitions, and also extend our main result to modified, bounded lattices for integer partitions with largest part $\leq m$ and number of parts $\leq$ n. Our main result states that:

$$\sum_{j=1}^{m(\lambda)} J(m_j(\lambda)) = J(\lambda) \cdot (s(\lambda) + 2)$$

where λ is a given partition, $m_k(\lambda)$ a partition covering λ, $s(\lambda)$ the number of partitions that λ covers and $J(\lambda)$ the partition interval between λ and $\varnothing$. For modified lattices this becomes:

$$\sum_{j=1}^{m(\lambda)} J(m_j(\lambda)) = J(\lambda) \cdot s(\lambda)$$

Lastly we show how our second major result, through the use of binary strings, can be given a computational interpretation and yield new insight into the behaviour of a random bubble-sorting algorithm – consequently enabling a new, more elementary, proof of the Lam conjecture. The paper was written with the intention of being understandable to a high-school student.

Contents

1 Introduction

Integer partitions have captivated such great mathematicians as Euler, Ramanujan and Hardy – yet, in their elementary incarnations, remain understandable to cavemen. The partitions of an integer are simply the different ways of writing it as a sum of positive integers. Take the number 3: its partitions are $1+2$, $1+1+1$ and 3 itself. However, beyond this deceivingly simple facade lie results of marvellous elegance and intriguing complexity; many of which arose due to the multiple possible geometrical and graph-theoretical representations of partitions, and representations and interpretations of the representations – and even further. As it turns out, a handful summands hold information about three-dimensional block constructions, the possible paths between two points, mosaic tilings and sorting algorithms.

This paper endeavours to prove several new identities in the theory of integer partitions. First we provide proper notation and definitions. Then we note how partitions can be classified in Young's lattice and proceed to prove one major theorem for the infinte unbounded lattice, and subsequently modify the same theorem into a flow identity for finite modified sublattices. We also describe the implications of our flow identity in the modified Young lattice, as it gives new insights into the mysterious behaviour of a random bubble-sorting algorithm on circular lists (cycles). Lam conjectured that the occurences of permutations of this list are linked to Pascal's triangle [1], and was proven correct by Aas in 2012 [2]. However, where Aas proof uses complicated Coxeter group theory, our identity seems to make a new, more elementary, proof possible. We briefly sketch the outlines of this application, but mainly refer readers to our sister paper by Johansen [3].

2 Standard Integer Partitions

A partition λ of an integer n consists of k integers

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_k), \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \tag{1}$$

summing to n, so that

$$n = |\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_k \tag{2}$$

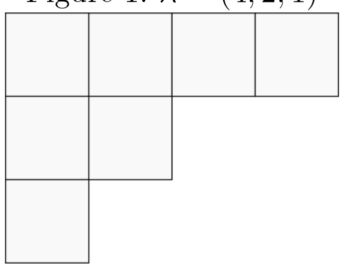
The number of possible partitions of n is denoted $p(n)$. For $p(n) = j$ we index the different partitions $\lambda^1, \lambda^2, ..., \lambda^j$, where $\lambda^1 = n$ and $\lambda^j = 1 + 1 + ... + 1$ (the exact conditions for ordering are specified in 2.1.2 Young's lattice and Hasse diagrams). Note specifically that by convention $p(0) = 1$.

It is necessary to differentiate between *partitions* and *compositions* – unlike the former, the latter takes the order of summands into account. That is, $1+1+2$ and $1+2+1$ are the same partition but different compositions. This paper will only concern partitions.

2.1 Graph-theoretical Representations

2.1.1 Ferrers Boards

Figure 1: $\lambda = (4, 2, 1)$

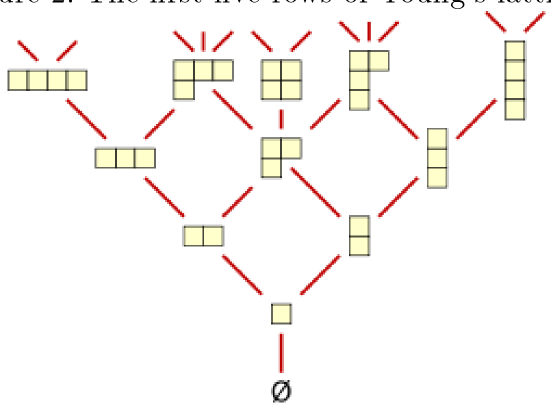


Partitions can be visualized using *Young diagrams*. The intuitive idea is to represent a partition λ of the number $|\lambda|$ as a group of $|\lambda|$ squares, where summands of size λ_k are represented by rows of length λ_k. See Figure 1 for an example where $|\lambda| = 7$, $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = 1$.

2.1.2 Young's Lattice and Hasse Diagrams

Partitions can be categorized into a partially ordered set, called *Young's lattice*. The lattice is commonly represented by a Hasse diagram, as shown in Figure 2. In the diagram, two partitions are connected if one can be obtained from the other by either adding or removing a single square. Starting from the empty set, those partitions created by adding a square to the first inner corner are placed in the rightmost position, those created by adding a square to the second inner corner are placed next to the rightmost position, etc.

Figure 2: The first five rows of Young's lattice

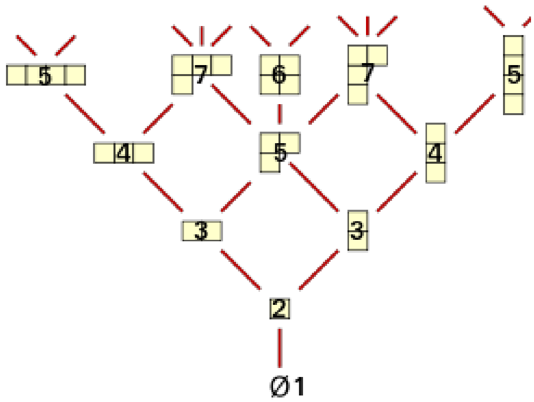


We index partitions in a row of the Hasse diagram from left to right, so that λ^1 is the leftmost partition, $\lambda^{p(|\lambda|)}$ the rightmost and λ^k the k^{th}. If a partition λ of size $|\lambda|$ and a partition μ of size $|\lambda| - 1$ are directly connected by a line in Young's lattice, λ *covers* μ. The partitions covered by λ we call its *slaves* and the partitions covering λ its *masters*. The number of slaves of λ we denote $s(\lambda)$ and the number of masters $m(\lambda)$, and in order to distinguish different slaves (or masters) we use the index $s_j(\lambda)$, $j = 1, 2, ..., s(\lambda)$, numbering the slaves from left to right (and likewise for the masters).

Given two partitions λ and μ such that $|\lambda| > |\mu|$, their *partition interval* in Young's lattice is the sublattice constituted by all partitions π such that $\mu \leq \pi \leq \lambda$.

The interval between λ and $\varnothing$ we call the *shadow* of λ, and the number of partitions constituting a shadow is its shadow-number, denoted $J(\lambda)$. (I have found a fairly effective – albeit large – formula for computing a shadow number knowing only $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, that has been left for Appendix A due to practical purposes.) Figure 3 shows the shadow-numbers for the partitions of the first five lattice rows.

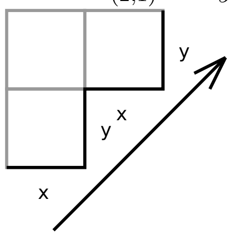
Figure 3: The first twelve shadow-numbers.



2.1.3 Paths and Words

The Young diagram need not be imagined as a stack of squares, but can instead be seen as a path connecting two points. If we trace a path from the lower left

Figure 4: $w_{(2,1)} = xyxy$.



corner to the upper right corner of a partition, only making right moves x and upward moves y, the shape of a partition can be described as a binary sequence of x's and y's, called a *word*. We denote the word of λ w_λ. For example, $w_{(2,1)} = xyxy$ and $w_{(1,1,1)} = xyyy$ (see figure 4). These entities will be crucial for outlining the implications of our second theorem in the field of computational combinatorics.

Seen as a path, an inner corner becomes yx and an outer corner becomes xy. Note specifically that inner corners need not lie within a partition – $w_{(2,1)} = xyxy$, that only appears to have one inner corner, can just as well be written $w_{(2,1)} = yxyxyx$ (the Young diagram remains identical) and in fact has three inner corners.

3 The Unbounded Young's Lattice

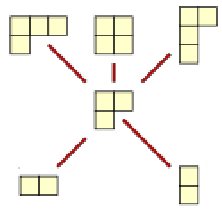
3.1 Preluding Theorems and Fundamental Properties

Theorem 1. *For every partition λ of Young's lattice,*

$$m(\lambda) = s(\lambda) + 1$$

Proof. The masters of a partition λ are the partitions that can be created by adding one square to λ without changing the positions of any other squares. Since we are considering partitions and not compositions, a square xy can only be placed in an inner corner yx. Thus, $m(\lambda) =$ the number of inner corners. Note now that a slave

Figure 5: The masters and slaves of $\lambda = (2, 1)$



is the opposite of a master, and reversing the master-creating process yields a slave of a partition. This is done by changing an outer corner xy into an inner corner yx. Thus $s(\lambda) =$ the number of outer corners. See Figure 5 for a visualization of the master/slave-creating processes. However, the inner and outer corners alternate throughout a partition, beginning and ending with an inner corner. Hence the number of inner corners will always be one more than the number of outer corners, and the result follows. □

Corollary 1. *Let* $\lambda^1, \lambda^2, ..., \lambda^k$ *be* k *partitions located next to each other in a row of Young's lattice. Then,*

$$\sum_{j=1}^{k} m(\lambda^j) = \sum_{j=1}^{k} s(\lambda^j) + k$$

Proof. By Theorem 1, $k = \sum_{j=1}^{k} 1 = \sum_{j=1}^{k} (m(\lambda^j) - s(\lambda^j)) = \sum_{j=1}^{k} m(\lambda^j) - \sum_{j=1}^{k} s(\lambda^j)$. Rearranging the terms yields the desired equality. □

Note that, for a given row of Young's lattice, the sum of the numbers of masters of all partitions is the same as the number of lines connecting that row to the row above. In the Hasse diagram it becomes apparent that the number of lines between two rows equals the sums of masters on the lower row, as well as the sums of slaves on the higher row, that is, given $|\mu| = |\lambda| - 1$:

$$\sum_{j=1}^{p(|\lambda|)} s(\lambda^j) = \sum_{j=1}^{p(|\lambda|-1)} m(\mu^j) \tag{3}$$

This observation yields a proposition concerning the relation between $p(|\lambda|)$ and $m(\lambda)$:

Theorem 2. *Let $\lambda^1, \lambda^2, ..., \lambda^{p(|\lambda|)}$ be an entire row of Young's lattice. Then*

$$\sum_{j=1}^{p(|\lambda|)} m(\lambda^j) = \sum_{j=0}^{|\lambda|} p(j)$$

Proof. The theorem trivially holds for $\lambda = \varnothing$ and $\lambda = (1)$. Consider now the cases for $|\lambda| \geq 2$. By setting $k = p(|\lambda|)$ in Corollary 1 and Eq. (3) we obtain the recursion $\sum_{j=1}^{p(|\lambda|)} m(\lambda^j) = \sum_{j=1}^{p(|\lambda|)} s(\lambda^j) + p(|\lambda|) = \sum_{j=1}^{p(|\lambda|-1)} m(\mu^j) + p(|\lambda|)$, where $|\mu| = |\lambda| - 1$. By setting $|\nu| = |\mu| - 1 = |\lambda| - 2$, and consecutively interating the process for additional variables, the expression can be expanded $\sum_{j=1}^{p(|\lambda|)} m(\lambda^j) = \sum_{j=1}^{p(|\lambda|-2)} m(\nu^j) + p(|\lambda|-1) + p(|\lambda|) = ... = \sum_{j=1}^{p(0)} m(0) + p(1) + \ldots + p(|\lambda|-1) + p(|\lambda|)$. Note now that $\sum_{j=1}^{p(0)} m(\varnothing) = m(\varnothing) = 1 = p(0)$, so that the rightmost side of the equation can be written $p(0) + p(1) + ... + p(|\lambda| - 1) + p(|\lambda|) = \sum_{j=0}^{|\lambda|} p(j)$. $\square$

3.2 A Flow Identity of the Unbounded Young lattice

Theorem 3. *For every partition λ of the unbounded Young lattice,*

$$\sum_{j=1}^{m(\lambda)} J(m_j(\lambda)) = J(\lambda) \cdot (s(\lambda) + 2)$$

In order to prove this theorem we shall introduce some additional definitions.

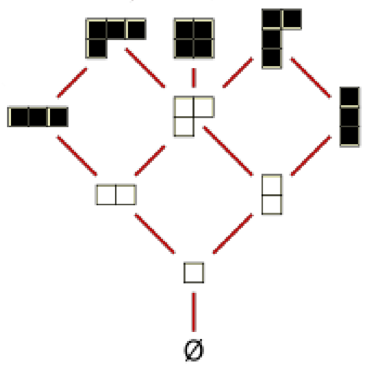
Definition 1. For a given shadow of λ, the *master structure* is the subset of Young's lattice including those partitions making up the shadows of the masters of λ, while excluding the shadow of λ.

Figure 6 demonstrates these definitions for $\lambda = (2, 1)$.

Definition 2. For a given partition λ, we denote by ${}_{\lambda}Y_j$ the vertical line of length y whose highest point is located in the inside of the j^{th} inner corner of λ (counted from left to right). When λ has been specified earlier, we use the shorthand ${}_{\lambda}Y_j = Y_j$.

If a partition μ has a square whose right side coincides with ${}_{\lambda}Y_j$ (when the two partitions are visualized atop each other in the plane), we say that μ *has* ${}_{\lambda}Y_j$. For example, let $\lambda =$ . Then $\mu =$  has ${}_{\lambda}Y_1$ and Y_2, while  only has Y_3.

Figure 6: The master structure (black) and the shadow of λ (white, including the empty set) for $\lambda = (2, 1)$.



We can now prove Theorem 3.

Proof. We will describe a bijection between the master structure and the shadow of λ. The reason for this is that when summing $J(m_j(\lambda))$, one will include the shadow of λ (or more properly, $J(\lambda)$) $m(\lambda)$ times, while including the master structure only once. And since Theorem 1 gives $s(\lambda)+2=m(\lambda)+1$, it remains to show that the number of partitions in the master structure is the same as the number in the shadow of λ.

Every partition in the master structure has a slave in the shadow of λ, created by removing one square from the the master partition. The bijection we use is to remove from each master partition the entire column in which this one square lies. Let us, before we give a general argument, demonstrate this bijection on a given partition $\lambda=(2,1)=\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}$. We see that the partitions to the upper left of the shadow of λ maps directly to their slaves, so that $m_1(\lambda)=\begin{smallmatrix}\square&\square&\square\\\square&&\end{smallmatrix}\mapsto\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}=\lambda$ and $s_1(m_1(\lambda))=$ $\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\mapsto\begin{smallmatrix}\square&\square\end{smallmatrix}$. As we progress to the right along the master structure, the bijection maps to the remaining partitions moving downwards along the shadow of λ. We have $m_2(\lambda)=\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\mapsto\begin{smallmatrix}\square\\\square\end{smallmatrix}$, $m_3(\lambda)=\begin{smallmatrix}\square&\square\\\square&\\\square&\end{smallmatrix}\mapsto\square$ and $s_2(m_3(\lambda))=\begin{smallmatrix}\square\\\square\end{smallmatrix}\mapsto\varnothing$.

For a partition λ with $m(\lambda)$ inner corners, the shadow of λ can be grouped into $m(\lambda)$ subsets as follows: the first set includes all partitions that have $Y_{m(\lambda)}$, the second those that have $Y_{m(\lambda)-1}$ but not $Y_{m(\lambda)}$, the third those that have $Y_{m(\lambda)-2}$ but not $Y_{m(\lambda)-1}$ and $Y_{m(\lambda)}$ etc., with the last set consisting of those partitions that have Y_1 but not Y_j, for $j=2,3,...,m(\lambda)$ (this set will also include $\varnothing$). Now $m_1(\lambda)$ will have a square whose left side coincides with $Y_{m(\lambda)}$ (recall the construction of the lattice in 2.1.2 Young's lattice and Hasse Diagrams), $m_2(\lambda)$ will have a square whose left side conincides with $Y_{m(\lambda)-1}$ and so forth; so in the same way we group the master structure into $m(\lambda)$ subsets, the only difference being that instead of having Y_k, $1\leq k\leq m(\lambda)$, these sets will have a block whose left side conincides with Y_k but none whose left side conicides with $Y_{k+1},Y_{k+2},...,Y_{m(\lambda)}$. In both groupings all elements of the same subset will be directly connected to each other by lattice lines. It is clear that the largest element in each set of the second kind will be a master of λ. This master then maps to the largest element in the corresponding set of the first kind - as our bijection removes the column in which a block was added, which is also the column in which there is a block whose left side coincides with Y_k, and hence the created partition cannot have a block's left side coninciding with Y_k but will instead have Y_k; and since the original partition by definition does not have any block whose left side coincides with $Y_{k+1}, Y_{k+2}, ..., Y_{m(\lambda)}$, a leftward shift of the columns located to the right of the removed column can never produce any block that has $Y_{k+1}, Y_{k+2}, ...,$ or $Y_{m(\lambda)}$ (which would have been illegal were the partition to be included in the given set). Consider then a master $m_k(\lambda)$ that has a square whose left side coincides with $Y_{m(\lambda)-k+1}$, and its corresponding subset. If $m_k(\lambda)$ has other slaves than λ, they will all have a square whose left side conincides with $Y_{m(\lambda)-k+1}$, since removing the square whose left side coincides with $Y_{m(\lambda)-k+1}$ would only yield λ itself. Among the slaves of those slaves, the ones having $Y_{m(\lambda)-k+1}$ will lie in the shadow of λ, as their Ferrers boards fit within the board of λ, while the others will lie in the master structure. Now let $m_k(\lambda)$ map to a partition μ in the shadow of λ. Then any slave of $m_k(\lambda)$ that is in the same set as $m_k(\lambda)$ (by the above definitions) will map to a slave of μ in the same set as μ. We have already seen why $m_k(\lambda)$ maps to the largest element of the second set. Then a slave of a partition is created by removing a block that forms an outer corner, and in order to stay within the first set we cannot remove the block with left side at Y_k. But the column in which this block lies is the only difference between $m_k(\lambda)$ and the partition it maps to, and hence removing any other block from $m_k(\lambda)$ will necessarily correspond to an identical operation on μ. Likewise we see that if a partition μ in the shadow structure maps to a partition in the master structure, the slaves of μ that lie in the same set will map to slaves in the same set as the master structure partition. The same argument can then be applied to the slaves of the slaves of $m_k(\lambda)$, etc. Thus, after consecutively applying the bijection to the partitions in the subset of the master structure with largest element $m_k(\lambda)$, we obtain all the partitions that have $Y_{m(\lambda)-k+1}$. Repeating this process for $k = 1, 2, ..., m(\lambda)$ maps the master structure to shadow of λ.

Consider once again, for clarification, the case of $\lambda = \ydiagram{2,1}$. First the shadow of λ is split into $m(\lambda) = 3$ subsets: elements of $\{\ydiagram{2,1}, \ydiagram{2}\}$ will have Y_3, elements of $\{\ydiagram{1,1}\}$ will have Y_2 but not Y_3, and elements of $\{\square, \varnothing\}$ will have Y_1 but neither Y_2 nor Y_3. Then the master structure is split into three corresponding subsets: $\{\ydiagram{3,1}, \ydiagram{3}\}$, $\{\ydiagram{2,2}\}$ and $\{\ydiagram{2,1,1}, \ydiagram{1,1,1}\}$. It is easily seen how these two collections of three sets map to each other.

$\square$

4 Limited Partitions

Suppose that one were to isolate all the partitions of Young's lattice that fit into a rectangle of given size, and from them construct a new lattice. This yields a *modified, bounded* lattice of *limited partitions.*

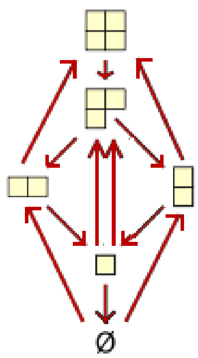
Definition 3. A (m, n)-limited partition is a partition with largest part $\leq m$ and number of parts $\leq n$.

In this modified Young lattice our previous theorems do not hold everywhere. However, by making certain adjustments we can reinvoke a new version of our main theorem. We begin by introducing flow; that is, giving each lattice line a direction. The lines that remain from the original lattice are turned into downgoing arrows, and we introduce additional upgoing arrows as follows:

Definition 4. There is an upgoing arrow from a partition μ to λ if λ can be obtained from μ by either adding a row m or a column n. This operation is equal to either placing the first letter of w_λ first or placing the first letter last.

Figure 7 shows the Young lattice for (2,2)-limited partitions. In the modified lattice, λ is a master of μ if there is an arrow from λ to μ, and λ is a slave to μ if there is an arrow from μ to λ. After adding the upgoing arrows, the graph is no longer a formal lattice (but instead a modified lattice, or *flow graph*). For each partition λ in the modified Lattice, *inflow* is the sum of shadow-numbers of its masters, and *outflow* is the shadow-number of λ multiplied by $s(\lambda)$. If inflow = outflow, *flow balance* holds.

Figure 7: The bounded modified Young lattice for $(2, 2)$-limited partitions.



4.1 A Flow Identity of the Bounded Modified Young Lattice

Theorem 4. *For a given partition λ in the bounded Young lattice,*

$$\sum_{j=1}^{m(\lambda)} J(m_j(\lambda)) = J(\lambda) \cdot s(\lambda)$$

That is, flow balance holds after adding the upgoing arrows.

Proof. Recall the proof of Theorem 3. For partitions with largest part $\leq m-1$ and number of parts $\leq n-1$, the same bijection still works. These partitions also have room for adding a row or a column, and hence receive two additional slaves through upgoing arrows, so that $s(\lambda)+2$ becomes $s(\lambda)$. However, as we are constrained by the measurements of rectangle $m \times n$, every partition with either a row m or column n will lose a larger master, and gain a slave (due to the row or column $\leq m, n-1$) – thus turning $s(\lambda)+2$ into $s(\lambda)$ – and every partition with both a row m and column n will lose two larger masters – thus also in this case turning $s(\lambda)+2$ into $s(\lambda)$. Note then that all partitions λ with a row m will be slaves to a partition μ such that $|\lambda| > |\mu|$ (by the definition of upgoing arrows). Furthermore, if $s_j(\lambda)$ also has a row m it will be a slave to $s_j(\mu)$. The same holds for the slaves of $s_j(\lambda)$ and so on, until for some partition ν, $s_j(\nu)$ no longer has a row m. But this is a one-to-one pairing between the exactly those partitions not given by the original bijection, and thus in itself suffices as a bijection for these special cases. □

5 Applications and a Proof of Lam's Conjecture

5.1 The Random Bubble-sorting Algorithm

Suppose that you were to sort a list of $n + 1$ elements (numbered from 0 to n) with the unusual characteristic of being circular – so that order cannot be established everywhere simultaneously. The random bubble-sorting algorithm randomly chooses two elements next to each other, switches their positions if they are unordered and leaves them untouched if they are ordered. Applied to a circular list over an infinite time span, it is impossible to determine the exact order generated by the algorithm (due to the randomness), but one can compute the probabilities of appearence of certain orderings.

Let us fix the element k, and name the other elements according to their relation to k. Numbers smaller than k we call y and numbers larger than k we call x. Now the different possible orderings of the list can be written as sequences of x's and y's – as words. If we convert our modified Young lattice, for (2,2)-limited partitions, into a set of words connected by arrows, we obtain a model for the flow of the sorting algorithm on a list of 5 elements 01...4. Note that we "fill out" all words to have the same length, namely 4, so that $w_{\varnothing} = yyxx$ and $w_(1) = yxyx$ etc. This can be done for any $n \in \mathbb{N}$. The process of switching places of two elements corresponds to removing a square, (xy becomes yx) and is thereby visualized through downgoing arrows. The upgoing arrows, by definition, corresponds to the operation of placing the first element last (or vice versa). Now, as the graph includes all the possible permutations of n-letter words, the shadow numbers of our flow identity correspond to number of possible intermediate steps as the sorting algorithm generates a specific permutation. This in turn correpsonds to the intermediate probabilities of the probability ratio between two orderings. A more detailed argument for this correspondence has been put forth by Johansen [3].

5.2 Lam's Conjecture

Thomas Lam, of the university of Michigan, conjectured in 2011 that the probabilities for the cycle permutations are linked to rows of Pascal's triangle:

Conjecture 1. *For a cycle with elements* $01...n$*, the probability ratio between the most likely and the least likely permutation is* $\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}$*, where* $\binom{n}{k}$ *is the binomial coefficient* $\frac{n!}{k!(n-k)!}$.

If the elements are $01\ldots n$ and the fixed element is k, the relative frequencies of occurence – that is, the specific probability ratios – are all in the flow graph under the $r \times k$-rectangle, where $r + k = n$. The full proof of this conjecture, although elementary, is too long to be included here, but can be found in the aforementioned paper by Emma Johansen.

6 Concluding Musings and Future Prospects

6.1 Theorems of Interest

There are many interesting properties of the Young lattice that are not relevant to the proofs of our main theorems, but may appear intriguing for the curious reader.

Theorem 5. *The number of upgoing arrows is the same as the total number of partitions.*

Proof. (Originally proved by Emma Johansen). Divide the (m, n)-limited partitions into three sets: those with all rows $\leq m-1$ and all columns $\leq n-1$, those with either a row m or a column n, and those with both a row m or a column n. We immediately notice that the second set receives one upgoing arrow for each partition, and the proposition holds in that case. Now the first set will have two upgoing arrows, and the third will have no upgoing arrows, so it remains to show that these sets are of equal size. If we observe the graph (Figure 7) we see that no partition in the first set will be a full $m \times n$, rectangle. If we first fill the remaining spaces of the rectangle, then remove the squares of the original partition, we obtain a partition from the third set. This correspondence is a consquence of the definition of the lattice, and as a bijection proves the theorem. □

There also appears to be a relation between shadow-numbers and slaves:

Conjecture 2. *For a given row $|\mu|$ of Young's lattice (standard or modified, but without arrows), let $s_{max}(|\mu|)$ be the number of slaves for the partition with the highest number of slaves. Then, for the modfied sublattice constituted by the shadow of a partition λ:*

$$J(\lambda) = 1 + \sum_{j=1}^{|\lambda|} s_{max}(j)$$

6.2 A Wide Array of Forward Paths

The properties of shadow-numbers are fairly unexplored. Rewarding prospects include the search for a generating function for shadow-numbers, the relations between $J(\lambda)$ and $p(|\lambda|)$, and perhaps a simpler formula for the shadow-number of a given partition.

Further opportunities abound when partitions are given new interpretations, as in the case with the bubble-sorting algorithm. At the same time extending and fusing mathematical fields, these situations may yield butterfly-effects of discoveries. However, by their very unexpected nature, such interpretations remain discoveries in the uncontrollable sense of the term; and a deliberate striving is likely to become an unproductively arduous endeavour. Optimally one keeps a focused discipline and an open mind, tracing worthwhile paths while remaining observant for promising tangents.

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References

[1] Lam, T. The shape of a random affine Weyl group element, and random core partitions. arxiv.org/abs/1102.4405.

[2] Aas, E. Stationary probability of the identity for the TASEP on a ring. http://arxiv.org/abs/1212.6366.

[3] Johansen, E. Random Bubble Sorting a Circular Bookshelf. Center for Computational Science and Communication, Royal Institute of Technology; 2013. http://www.raysforexcellence.se/wp-content/uploads/2013/01/Emma-Johansen-Random-Bubble-Sorting-of-a-Circular-Bookshelf.pdf.

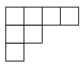
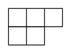
A A formula for shadow numbers

Instead of writing $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_k, \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$, let us write $|\lambda| = \alpha_1\lambda_1 + \alpha_2\lambda_2 + ... + \alpha_l\lambda_l$, where $\lambda_1 > \lambda_2 > ... > \lambda_l$, $\alpha_1 + \alpha_2 + ... + \alpha_l = k$, and l is the number of distinct parts of λ. (Note specifically that $l = s(\lambda)$.) For a given partition this notation becomes $\lambda = (\alpha_1\lambda_1, \alpha_2\lambda_2, ..., \alpha_l\lambda_l)$, and is a way of simplyfying the notation by compressing larger rectangles $\alpha_j\lambda_j$ into a single term, instead of having to repeat each row λ_j of the rectangle α_j times.

Theorem 6. *Let $\lambda = (\alpha_1\lambda_1, \alpha_2\lambda_2, ..., \alpha_l\lambda_l)$, $l = s(\lambda)$, be a partition in Young's lattice. Then:*

$$J(\lambda) = \sum_{k=0}^{\lceil \frac{l}{2} \rceil} \binom{\alpha_{k+1} + ... + \alpha_{l-k} + \lambda_{k+1}}{\lambda_{k+1}} - \sum_{k=0}^{\lceil \frac{l}{2} \rceil} \binom{\alpha_{k+2} + ... + \alpha_{l-k} + \lambda_{k+1} - \lambda_{l-k}}{\lambda_{k+1} - \lambda_{l-k}}$$

where $\binom{\alpha}{\lambda}$ is the binomial coeffeficient $\frac{\alpha!}{\lambda!(\alpha-\lambda)!)}$.

Proof. For a partition λ that does not form a perfect rectangle $(\alpha_1+\alpha_2+...+\alpha_l)\times\lambda_1$, let λ^* be the partition that together with λ forms such rectangle (after eventual rotation in the plane). E.g., if $\lambda =$  $= (4,2,1)$ then $\lambda^* =$  $= (3,2)$. Generally, we have $\lambda^* = \Big(\alpha_l(\lambda_1 - \lambda_l), \alpha_{l-1}(\lambda_1 - \lambda_{l-1}), ..., \alpha_2(\lambda_1 - \lambda_2)\Big)$. Denote the rectangle $(\alpha_1 + \alpha_2 + ... + \alpha_l) \times \lambda_1$ by $\lambda + \lambda^*$. Now we make the crucial observation that $J(\lambda) = J(\lambda+\lambda^*) - J(\lambda^*)$. We reach this conclusion as the interval between λ and $\lambda+\lambda^*$ includes $\lambda, \lambda+\lambda^*$, and all partitions formed by fitting smaller partitions into the "hole" in λ, which is $J(\lambda^*)-2$ partitions (excluding λ and $\lambda+\lambda^*$), so that $J(\lambda+\lambda^*) - J(\lambda) = J(\lambda^*)$. Whenever $\lambda = (\alpha_1\lambda_1, \alpha_2\lambda_2)$, the determining of $J(\lambda)$ is straightforward. For partitions that are composed of more than two rectangles, $J(\lambda^*)$ can be calculated as $J(\lambda^*) - J(\lambda^{**})$, etc. As we know how to determine λ

and λ^*, we get $\lambda^{**} = \Big(\alpha_2\big(\lambda_1 - \lambda_l - (\lambda_1 - \lambda_2)\big), \alpha_3\big(\lambda_1 - \lambda_l - (\lambda_1 - \lambda_3)\big), ..., \alpha_{l-1}(\lambda_1 - \lambda_l - \big(\lambda_1 - \lambda_{l-1})\big)\Big) = \Big(\alpha_2\big(\lambda_2 - \lambda_l\big), \alpha_3\big(\lambda_3 - \lambda_l\big), ..., \alpha_{l-1}(\lambda_{l-1} - \lambda_l)\Big)$. Generalizing these results, we get, for even $2j, 0 \leq j \leq \lceil \frac{l}{2} \rceil$:

$$\lambda^{2j*} = \Big(\alpha_{j+1}(\lambda_{j+1} - \lambda_{l-j+1}), ..., \alpha_{l-j}(\lambda_{l-j} - \lambda_{l-j+1})\Big)$$

and for odd $2j+1$:

$$\lambda^{(2j+1)*} = \Big(\alpha_{l-j}(\lambda_{j+1} - \lambda_{l-j}), ..., \alpha_{j+2}(\lambda_{j+1} - \lambda_{j+2})\Big)$$

This gives: $\lambda^{2j*} + \lambda^{(2j+1)*} = \big((\alpha_{j+1} + ... + \alpha_{l-j})\lambda_{j+1}\big)$, and $\lambda^{(2j+1)*} + \lambda^{(2j+2)*} = \big((\alpha_{j+2} + ... + \alpha_{l-j})(\lambda_{j+1} - \lambda_{l-j})\big)$, which yields the following shadow-numbers: $J(\lambda^{(2j)*} + \lambda^{(2j+1)}) = \binom{\alpha_{j+1} + ... + \alpha_{l-j} + \lambda_{j+1}}{\lambda_{j+1}}$ and $J(\lambda^{(2k+1)*} + \lambda^{(2k+2)*}) = \binom{\alpha_{j+2} + ... + \alpha_{l-j} + \lambda_{j+1} - \lambda_{l-j}}{\lambda_{j+1} - \lambda_{l-j}}$ (according to the binomial theorem). Lastly, as above deduced, the shadow numbers can be computed:

$$J(\lambda + \lambda^*) - J(\lambda^* + \lambda^{**}) + ... + J(\lambda^{2j*} + \lambda^{(2j+1)*}) - J(\lambda^{(2j+1)*} + \lambda^{(2j+2)*}) + ... =$$

$$\sum_{k=0}^{\lceil \frac{l}{2} \rceil} J(\lambda^{(2k)*} + \lambda^{(2k+1)*}) - \sum_{k=0}^{\lceil \frac{l}{2} \rceil} J(\lambda^{(2k+1)*} + \lambda^{(2k+2)*})$$

Inserting the binomial formula for shadow numbers yields the desired result. $\square$