Tubings and Nestings: Where Graphs and Line Graphs Meet

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Abstract

In this paper we show that the poset (partially ordered set) formed through a method of associating elements in a graph called nesting, corresponds to a combinatorially unique convex polytope. This was inspired by previous results, showing the same correspondence for a combinatorially distinct method of associating elements called tubing [?]. This result was obtained by using results from graph theory by H. Whitney (1932) and L.W. Beineke (1970) to find a bijection between the poset of nestings of a given graph Γ and the poset of tubings of another graph, uniquely determined by Γ.

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Contents

Glossary

K_n The complete graph with n vertices.

$K_{n,m}$ The bipartite graph with one group of n and m vertices respectively.

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Associahedron The polytope that embeds an algebra associative up to homotopy.

Bijection A function that is both surjective and injective..

circuits/cycle A trail which visits a vertex twice.

Codimensionality The difference in dimensionality between objects. Codimension 1 is equivalent to one dimension lower than the reference object.

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Cyclohedron The convex polytope generated by the cyclic group.

Equivalence class A partitioning of elements by an order relation that is transitive, reflexive and symmetric.

Face Poset The poset of cells in a polytope of different dimensions. Cells are related if one is contained in the other.

Graph Associahedron the polytope embedded in the poset created by a all possible tubings of a given graph.

Group A set and a binary associative operation such that the set is closed under the operation Every element also has an inverse under the operation and the set contains a neutral element.

Homotopy A continuous deformation/morphism.

Isomorphism A morphism that is bijective.

N-dimensional Cell The n-dimensional face. For example, the 0-dimensional face is a point, the 1-dimensional cell is an edge and so on.

Nullity The nullity of a graph is a number given by: $m - n + c$, where m is the number of edges, n the number of vertices and c the number of connected components.

Partially ordered set (poset) A set with an order relation that is not necessarily defined between all pairs of elements in the set.

Partition A division of the elements of a set, such that all elements belong to exactly one group.

Path graph An undirected and unordered graph containing two elements that are only connected to one edge, and all other elements lie on the path between these to nodes.

Permutahedron The convex polytope generated by the complete graphs with respect to tubings and the biparte graph $K_{1,n}$ for nestings.

Polytope The intersection in n dimensions of at least n+1 half spaces.

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Trail An ordered collection of edges, in which all consecutive edges are adjacent.

1 Introduction

Within mathematics, the concept of graphs is both useful and illuminating in areas ranging from topology and homotopy theory to combinatorics and geometry. Graphs and graph theory are also central to several applied fields such as computer modelling, data/network analysis and more recently neural networks. In all of these instances, the ability to compare the structure of graphs is fundamental.

One way in which this can be done, is through associating components of a graph. A partially ordered set (poset) is then obtained from the set of different associations of the components in the graph. Thus the set of graphs can be divided into equivalence classes, within which graphs share their combinatorial structure of components and component-relations.

One way of partitioning graphs in this way, is through a method called tubing. This method has been shown to be particularly useful, since the tube poset has been proven to be isomorphic to the face poset of a combinatorially unique convex polytope. Such polytopes are called *graph associahedra* [1, 2].

Another way of partitioning graphs is inspired by the work on homotopic associativity by D. Tamari and J. Stasheff, in which they bracked strings of letters [3, 4]. We will interpret this as the nesting of path graphs, which can then be generalized to a method of nesting for an arbitrary graph. It should be noted, that this method is not combinatorially equivalent to tubing. The aim of this paper is then to provide a comparison between the methods of nesting and tubing. In particular, we will show the following parallel between the two methods.

Theorem 1. The poset of nestings of a given simple graph Γ, is the face poset of a combinatorially unique polytope.

This theorem is proven, by first showing that for a given graph Γ, there exists an unique graph -called the line graph of Γ- with a poset of tubings identical to the poset of nestings of Γ. As mentioned, the tube-poset of all graphs has already been shown to correspond to the face poset of a graph associahedron, and theorem 1 therefore follows.

2 Definitions and notation

Definition 1. *(Graph)* A *graph* $\Gamma = (V, E)$ is a finite set V called the *vertices* and $E \subseteq V \times V$ called the *edges*. In this paper, a graph is also considered to be connected and free of repeated edges and loops. Thus, each unordered pair $(u, w) \in V \times V$ has at most one representation in E, and $(u, u) \notin E$ for any pair (u, u).

In the following definitions, Γ is defined as a graph with the set of vertices V and edges E.

Definition 2. *(Subgraph)* Let $\bar{V}$ denote a subset of V and $E_{\bar{V}} \subset E$ denote all pairs $(\bar{u}, \bar{w}) \in \bar{V} \times \bar{V}$. Then a *subgraph* $\bar{\Gamma}$ of Γ is defined as a set of vertices $\bar{V} \subset V$ and edges $\bar{E} \subseteq E_{\bar{V}}$ such that $\bar{\Gamma}$ is a graph. A subgraph is *filled* if $\bar{E} = E_{\bar{V}}$.

Example 1. See Figure 1.

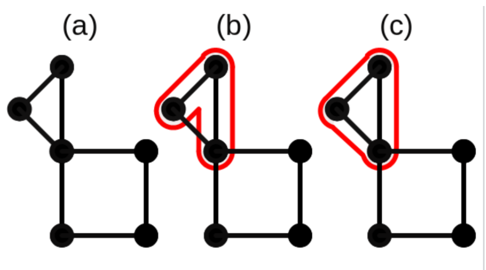

Figure 1: (a) a graph Γ, (b) a subgraph $\bar{\Gamma}$ of Γ, and (c) a filled subgraph $\bar{\Gamma}_{max}$ of Γ.

Definition 3. *(Tubing)* A *tube* in Γ is a subgraph of Γ containing at least one vertex.

A *k-tubing* is a set of k tubes in Γ, that do not intersect and are not adjacent.

Tubes *intersect* if they share a vertex, but neither is contained in the other.

Two tubes are *adjacent* if they have no vertex in common, but there exist two vertices in the respective sets that are connected by an edge.

Definition 4. *(Nesting)* A *nest* in Γ is either (1) a subgraph of Γ containing at least two vertices or (2) a graph $T = (V, E')$ where $V(\Gamma) = V(T)$ but $E' \subset E$.

A *k-nesting* is a set of k nests in Γ such that no pair of nests intersect.

For the rest of the paper, let $N(\Gamma)$ and $T(\Gamma)$ denote the set of all possible nestings and tubings of Γ respectively.

Remark. (*Distinction between tubings and nestings*) One useful interpretation, is to view tubings as a an operation on the vertices and nesting as an operation on the edges.

This interpretation provides intuition to why tubings can consist of singletons (isolated vertices) but nestings always contains an edge. But also to why a tube always contains a proper subset of the vertices of the graph, whereas a nest always consists of a proper subset of the edges of a graph.

In Figure 2 a maximal nesting and tubing of a path graph is shown. Path graphs are a family of graphs, for which the relationship between tubings and nestings are particularly simple. The reason for this is further illuminated in later sections.

Figure 2: A maximal (a) tubing and (b) nesting of a path graph.

3 Correspondence between nestings and tubings

Definition 5. *(Line Graph)* For every graph Γ, there exist a corresponding line graph Γ_L). The graph Γ_L is constructed through a bijection between $E(\Gamma)$ and $V(\Gamma_L)$.[1]

This bijection conserves adjacency, nullity and circuits, and is thus uniquely defined [5].

Note: For disconnected graphs, repeated edges and loops Γ_L is not uniquely defined, but these cases are examined in section 4.

[1]Here, $V(\Gamma)$ denote the set of vertices of a graph Γ and $E(\Gamma)$ the set of edges.

Example 2. The procedure for constructing a line graph is illustrated in Figure 3. The inverse procedure is discussed in section 4.

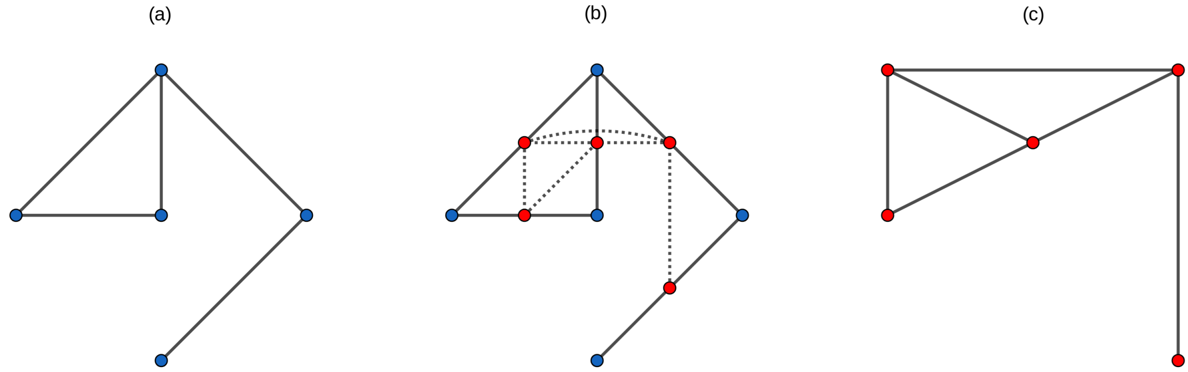

Figure 3: A transformation from (a) a graph Γ to its (c) line graph Γ_L. (b) shows how every vertex in Γ_L correspond to an edge in Γ and every adjacency of edges in Γ correspond to an adjacency of vertices in Γ_L.

Theorem 2. For every graph Γ, there is a corresponding graph Γ_L, in which the set of nestings of Γ is in bijection with the set of tubings of Γ_L. In particular there exists a bijection such that: $N(\Gamma) \xrightarrow{L} T(\Gamma_L)$. for a given Γ[2]

Proof. Under L, every edge in Γ is transformed into a distinct vertex in Γ_L. Two vertices in Γ_L are then connected by exactly one edge, if and only if their preimage in Γ are connected by a vertex. This procedure cannot connect vertices whose preimages were not adjacent. Therefore, L is well defined.

Now consider the nestings of Γ. All edges belonging to a nest will be transformed into vertices with the same correspondence. Therefore, the transformations of single nests are single tubes. We must now show that nestings in Γ correspond to valid tubings in Γ_L. That is, the image of two nests in Γ cannot (i) intersect or be (ii) adjacent.

(i) Assume that two tubes in Γ_L intersect. Then the corresponding nestings in Γ share one edge, and thereby intersect, which goes against the construction of the nesting in the preimage.

[2]Γ is fixed by necessity, as will become clear in the section about the inverse correpondence.

(ii) Assume instead that two tubes in Γ_L are adjacent. Then, there were two distinct nests in the preimage containing adjacent edges. Since the edges are always bounded by vertices, both nests would have contained the vertex shared by the aforementioned edges. Again, the nests would have intersected, which is contrary to their construction.

Since the transformation of tubings is simply the transformation of a set of marked vertices, it follows that : $N(\Gamma) \to T(\Gamma_L)$ for a given Γ. $\square$

4 The inverse correspondence

Following the results from theorem 2, the inverse relationship comes into question, as well as the injectivity and surjectivity of L.

Theorem 3. (Whitney's isomorphism theorem) L is injective with the exception of two graphs, K_3 and $K_{1,3}$.

Proof. Two graphs are considered to be identical, if their cycles, nullity and edge/vertex adjacency are in bijection.

Since the line graph inherits these properties, the statement of injectivity is equivalent to the following statement: two graphs Γ_1 and Γ_2 are isomorphic if and only if $\Gamma_{L1}) \cong L(\Gamma_{L2})$.

This is *Whitney's isomorphism theorem* and was proven in 1932 [5], with two exceptions: K_3 and $K_{1,3}$ both of which are mapped onto K_3 under L.

Note: there is still a bijection between (Γ) and $T(\Gamma_L)$ for a fixed Γ. $\square$

Theorem 4. (Beineke's theorem) L is not surjective, but the family of graphs not in the range of L is easily characterized.

Proof. As proven by L.W. Beineke in 1970 [6], a given graph Γ is a line graph if and only if Γ does not contain one of nine forbidden graphs as a filled subgraph.

Consequently, infinitely many graphs are not in the range of L, but all of these graphs belong to a very specific family of graphs. $\square$

Observation 1. We therefore conclude that $L : \Gamma \to \Gamma_L$ is a bijection, where the domain is all graphs $\Gamma : \Gamma \ncong K_3, K_{1,3}$ and the range of L is all graphs $\Gamma : \Gamma_{not} \not\subset \Gamma$, where Γ_{not} denotes any of the forbidden subgraphs. Therefore, $\Gamma \to \Gamma_L$ a bijection over this modified domain and range. Therefore the weaker proposition, the isomorphism of nestings and tubes of an arbitrary graph follows.

Remark. All results in this paper have been shown for graphs with restrictions on connectivity, repetitive edges and the directions of edges. Such graphs are sometimes called simple graphs, whereas graphs without these restrictions are called disconnected graphs, multigraphs and directed graphs. This terminology will be used in the following remark, where we investigate the action of L on non-simple graphs.

Disconnected graphs Following the results for simple graphs, it is quickly realized that the line graph is uniquely defined for disconnected graphs as well, and the line graph thus continues to be well defined. However, the inverse correspondence becomes more problematic. This is realized by the fact that for a given graph Γ and its line graph, an arbitrary amount of singletons can be added to Γ, without changing the corresponding line graph. One should note however, that even though adding singletons changes the graph, it does not change the possible nestings of the graph.

However, if the disconnected components in Γ are not singletons, then the previous results for simple graphs apply to the individual components, and thus also for the collection of disconnected components.

Loops Loops gives rise to similar problems as disconnectivity does, as illustrated in Figure 4, where both graphs have the same line graph. The line graph of every graph is still well defined, but again, the inverse correspondence is not well defined. One should note yet again, that the nesting poset is unchanged by a transformation between the graphs in Figure 4.

Repeated edges For repeated edges however, one must define how vertices whose preimages share two vertices, are mapped under L.

Directed graphs The graphs with directed edges is simply a special case of the graphs

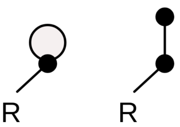

Figure 4: Multigraphs which give the same line graphs

considered in all previous sections. How they are affected by L is simply determined by their other graph properties such as connectivity, repetition of edges and whether they contain loops.

5 Examples

We have now established that both the nesting and tubing poset of a graph defines the face poset of a convex polytope. To showcase the differences and similarities between graphs and their line graphs as well as the nesting and tubing posets, the following section will provide some illustrated examples. We will particularly showcase families of graphs where such correspondences are simple.

5.1 Cyclic graphs

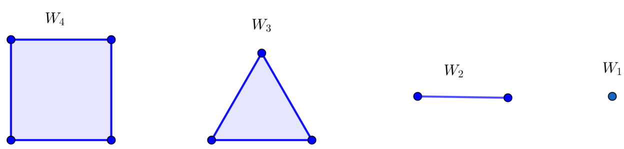

Figure 5: The first four cyclic graphs

The cyclic graphs, denoted by W_n, are illustrated in Figure 5. These are a particularly simple family of graphs for which the graphs and line graphs are isomorphic, i.e. $\Gamma \cong \Gamma_L \to N(\Gamma) = T(\Gamma_L) = T(\Gamma)$. This is shown in 6.[3] Therefore, the polytopes realized by the nestings and tubings of a cyclic graph are combinatorially identical. The polytopes

[3]In fact, this property of being stable under L, such that the number of edges or vertices do not decrease toward zero or increase toward infinity is very rare. It was proven by Bienkene (1970) to be true only for graphs in which the degree of every vertex is two.

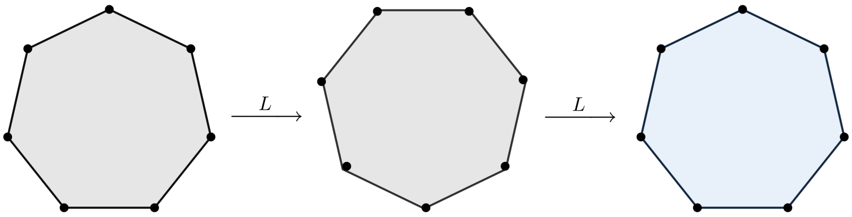

Figure 6: As shown in the figure, the action of L on W_n is identical to the action of the identity transformation.

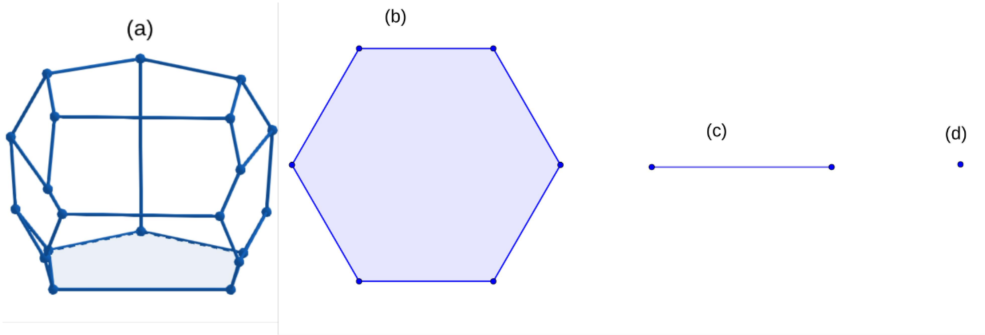

Figure 7: The polytopes corresponding to the poset of nests and tubes of (a) W_4 (b) W_3 (c) W_2 and (d)W_1

generated by these nesting and tubing posets are called cyclohedra, and are illustrated in Figure 7.

5.2 Path graph

A path is a cycle, with one edge removed.These graphs are denoted P_n in this paper, and the first three such graphs are reproduced in Figure 8

In every iteration of L, the path will therefore, in contrast to cycle graphs, become one node and edge shorter but still belong to the same family of graphs, also shown in Figure 8.

Figure 8: The action of L on P_n

This means that the poset of nestings of P_n realizes a polytope of codimension 1 to the polytope realized by the poset of tubings of P_n. This is also realized, by noting that $N(P_n) = T(P_{nL}) = T(P_{n-1})$. The family of polytopes generated by the poset of tubings and nesting of P_n is the family called associahedra.The first four associahedra are illustrated in Figure 9, where the polytopes corresponding set of tube/nest posets are also given.[4]

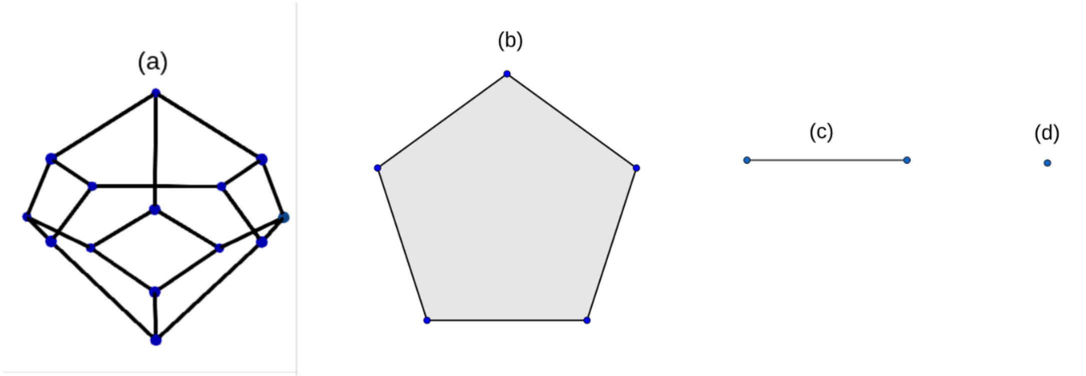

Figure 9: The polytopes generated by (a) $N(P_4)$ and $T(P_3)$ (b)$N(P_3)$ and $T(P_2)$ (c) $N(P_2)$ and $T(P_1)$ and (d) is arbitrarily defined

5.3 Permutation graphs

Of course, not all graphs have such a simple graph-line graph correspondence. An example of this is the permutation graphs. These are the graphs whose tube/nest posets correspond to all possible permutations of the vertices/edges in a given graph. These graph generate a family of polytopes called *permutahedra*. For tube posets, the permutation graphs are the symmetric graphs. As expected, the line graph of the permutation graph (with regards to nesting) is a symmetric graph. This is shown in Figure 10, where it also becomes clear that the line graph of a symmetric graph not has the same correspondence to its preimage as previous examples.

[4]The associahedra generated by the tubing of P_n is denoted K_n, which is not to be confused with the complete graph with n vertices, also denoted by K_n

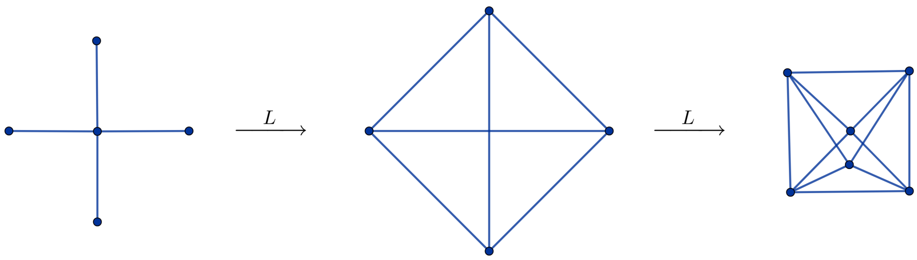

Figure 10: The action of L on the permutation graphs

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